Instantaneous Measurement of Nonlocal Variables

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August 2006

Abstract

We ask the question: is it possible to measure nonlocal properties of a state instantaneously? Relativistic causality restricts how this can be done, but we show here, for the bipartite two dimensional case, that every nonlocal variable of the system can be measured instantaneously. A necessary condition for a general nonlocal measurement on a bipartite system is presented.

This essay was written and submitted as part of the University of Cambridge's Part III Mathematics in 2006. Many thanks to Berry Groisman for being a helpful supervisor. Corrections have been added since submission.

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1 Introduction

In order to grant variables the status of observables, we must show that they can be defined at any instant. Thus we must show that they can be measured instantaneously. In a relativistic theory, this presents problems that have only recently been solved.

In this essay, we will solve this problem for the simple system of two particles, each with a two dimensional Hilbert space, for example the spin state of two spin-1/2 particles. The goal is to construct some procedure that can determine which of four orthogonal states the system is in[1]. We will in fact find that such a method does exist so every operator is measurable instantaneously so all operators are observables. However, we will find causality imposes restrictions on how this can be done and on the final state. We will show that the normal measurement process of quantum mechanics, that leaves the system in a eigenstate of the operator, cannot in general be carried over to a relativistic theory.

1.1 Localised Particles

It is clear that, theoretically at least, we can perform any collective measurement of any number of particles by interacting them in some way. When we have all the particles in the same small region, we can perform global unitary transforms and collective measurements in an arbitrarily small time. So any operator is measurable when the particles are together thus all variables are observable. As an example, we show how the four maximally entangled states, known as Bell states, can be distinguished. The circuit in Fig. 1 maps each Bell state onto a unique product state. Note it uses a controlled-NOT gate which is a *global* unitary transform. The four product states can be distinguished by measuring the spin, polarisation or whatever other physical variable we encode our information in for each particle. So by distinguishing the four product states, we have distinguished the four Bell states. If we require the system to be undisturbed, we can put the particles through the reverse circuit (i.e. Hermitian conjugate of the unitary operator) to bring the state back to the initial Bell state.

A natural question to ask is how far can we extend the above if the particles are separated by long distances, so only *local* operations can be applied. Can we perform some measurement to find their joint (entangled) state? Can we do this without disturbing the state? These questions will be answered and the severe restrictions imposed by causality will be discussed. We will relax the restrictions on

[1] If the system is not in one of these states the usual probabilistic rule of quantum mechanical measurement applies.

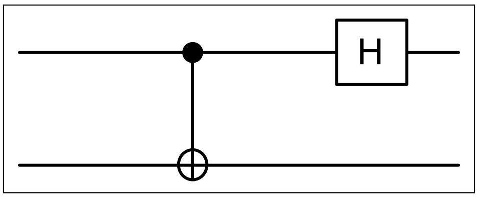


Figure 1: Quantum circuit to distinguish the four maximally entangled states. The gates are Controlled-Not and the Hadamard gate (Eqn. 16).

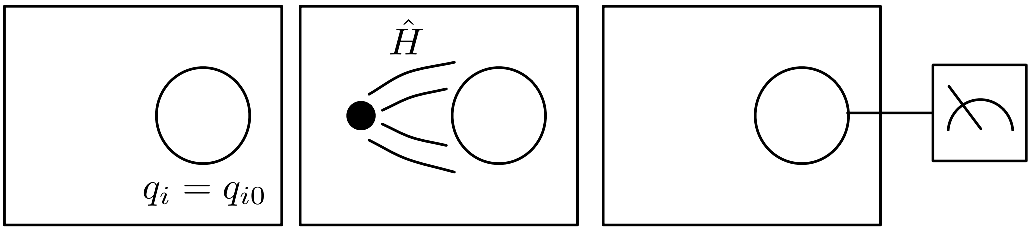
the post measurement state as the essay progresses and find a corresponding increase in what becomes measurable.

2 Measurement

Here we explain the measurement process of quantum mechanics in detail so we can understand how to perform a nonlocal measurement.

2.1 von Neumann Measurement

The standard measurement in quantum mechanics was described by von Neumann [1] as having three distinct parts. The purpose of a quantum measuring device is to interact with the quantum system to be measured in some way, so it becomes entangled with it. Then some classical system reads off the state of the measuring device. So if the measuring device has n degrees of freedom, for example the positions of dials, we can define n operators q_i to be the corresponding observables. The classical system measures these, after interaction with the system. For example, we can consider a Stern-Gerlach experiment. A spin-1/2 particle passes through a magnetic field and bends up or down, depending on the direction of the spin. So the single degree of freedom of the measuring device is in fact the position of the particle some fixed distance after the magnetic field. When the particle strikes the screen, which we regard as a classical device, the position is measured and the measurement of the spin is complete.



(a) Measuring device state preparation

(b) Interaction

(c) Classical measurement

Figure 2: (a) Preparing the state of the measuring device. (b) Interaction with the system. (c) Classical measurement of the measuring device state.

We will need to formalise the above to use in the construction of a nonlocal measurement.

So in the first part of the process, the measuring device is prepared in some initial state (Fig. 2(a)). This corresponds to setting the observables q_i to some initial values. Next, we allow the device to interact with the system (Fig. 2(b)) via some Hamiltonian $H_{int} = \Sigma_i g(t) q_i A_i$, where $g(t)$ is some coupling in time and A_i are observables of the system we want to measure. So each degree of freedom q_i measures an observable A_i so there must be a different eigenstate of q_i corresponding to each eigenstate of A_i in order to measure A_i completely. Since we are dealing with an instantaneous measurement, $g(t)$ is only non-zero for a short time. For future convenience, we further normalise $g(t)$ such that $\int g(t)dt = 1$ [2].

So now the system evolves according to the Schrödinger equation and we end up, in general, with the measuring device entangled with the system. So now, finally, we measure the coordinates of the measuring device with a classical system (Fig. 2(c)). It is only at this point that the collapse occurs and there is a definite outcome of the measurement. Because of the entanglement, the system also collapses into some state. We can see that this is in fact an eigenstate of the A_i (assuming they commute), since the A_i commute with H_{int} so do not change. So a von Neumann measurement is the standard measurement of quantum mechanics and is the kind of measurement presented in Refs. [2, 3].

2.2 Other Measurement Types

However, other less restrictive types of measurement are possible. Ref. [2] proposes the terms *operator specific* and *state specific* measurements. An operator specific measurement is that just discussed; we measure a value of some operator(s) i.e. project the state onto the eigenstates of some operator. However, a state specific measurement is more basic, as it just verifies that the system is in a certain state. The outcome is simply a "yes" or "no".

There is also a distinction between measurements that leave eigenstates of the operator undisturbed (*nondemolition*) and ones that do not (*demolition*). In the case of a state specific measurement, a nondemolition measurement leaves the system undisturbed if the answer was "yes" but need not if the answer was "no". So other eigenstates may be disturbed. A demolition verification measurement does not even require the state to be undisturbed if the answer was "yes". We shall see that restrictions on the final state, together with causality, give the restrictions on what can be measured.

We note that any von Neumann measurement of an observable can be considered a set of verification measurements of the eigenstates of that observable, with constraints on the final states to ensure all eigenstates undisturbed. So anything we show for a general verification measurement will be true for all measurements.

In this essay, we will work through these three types of measurement in turn, getting less restrictive and consequently more observables become measurable. We will firstly discuss operator specific measurements of nonlocal variables, and find that only essentially one nonlocal variable is measurable. Then we move on to state specific measurements and see that more states can be verified, but this will in general disturb other eigenstates. Finally, we discuss demolition operator specific measurements and see that every state becomes measurable. Note that we restrict attention to bipartite systems, with each subsystem Hilbert space of dimension 2.

3 Quantum Mechanics and Special Relativity

Throughout the remainder of this essay, we will use the characters *Alice* and *Bob* to carry out the measurement. We will assume they are separated by some large distance x so that anything that happens in time less than x/c can be regarded as 'instantaneous' for the purpose of discussing causality.

3.1 Instantaneous Measurements

Using the above definition of instantaneity, we can define what we mean by an instantaneous measurement. Clearly it does not mean that the result of the measurement is known instantaneously since this would require instantaneous communication of the results of local measurements between Alice and Bob. What we do mean, however, is that the results of measurements are recorded *classically* and instantaneously. It is important that it is classical for the result to actually exist at that time, even if no observer can know it.[2] Then finite time is required to communicate the partial results so that the final result is known. So we require that all interactions with the particles are instantaneous, that is, they take a finite time ϵ which is independent of the distance between Alice and Bob. Therefore the collapse to the final state has occurred irreversibly (independent of whether the final result is actually known at any point) and any later interaction with the particles will not cause the result to change.

3.2 Local Operations

We can show straightforwardly that local operations in quantum mechanics do not give violations of causality. *A priori* this is not apparent, since quantum mechanics is a nonlocal theory.

For causality to be obeyed, the probabilities of outcomes for any measurement performed by Bob must be independent of anything Alice performs. The density matrix formalism is the natural language for this situation; states can be distinguished by some measurement if and only if their density matrices are different. Note that we will only need to consider unitary transformations applied by Alice. This is because any measurement or interaction with another system (belonging to Alice) can be considered just a unitary evolution of the joint (Alice's) system, if the result is not known. Clearly Bob cannot know the outcome of any measurement so from his point of view there is no collapse.

If Alice performs a unitary transform u_A, the joint density matrix ρ becomes

$$\rho \to \rho' = u_A \otimes I_B \rho u_A^\dagger \otimes I_B$$

where I_B is the unit operator acting on Bob's particle. Performing the partial trace to get Bob's state we find

$$\begin{aligned} \rho_B &= \mathrm{Tr}_A(u_A \otimes I_B \rho u_A^\dagger \otimes I_B) \\ &= \mathrm{Tr}_A(u_A^\dagger \otimes I_B u_A \otimes I_B \rho) \\ &= \mathrm{Tr}_A \rho \end{aligned}$$

which is clearly independent of u_A.

3.3 Nonlocal Measurement

We have just seen that local unitary operations and therefore local measurements cannot violate causality. If the possibility of a nonlocal measurement is included we find a restriction on the nonlocal measurement so that causality is obeyed [4].

Again, we require that whatever Alice does before the nonlocal measurement, Bob gets the same outcomes with the same probabilities after the measurement. So before the nonlocal measurement is carried out at time t_0, Alice can perform any unitary transformation on her system, at time $t_0 - \epsilon$. So if the initial

[2] If the result was recorded in some quantum state, to be measured later, then in general the result can be one of a number of possibilities with some probability distribution. So clearly in this case we cannot say the measurement is complete since the result still has not been chosen from the possibilities.

state is $|\psi\rangle$, and the unitary operation carried out by Alice is u_A, the state at time t_0 is $u_A \otimes I_B |\psi\rangle$. At time $t_0 + \epsilon$, after the nonlocal measurement, Bob measures a local observable A_B and gets outcome a. So causality in this case gives the restriction

$$p(\psi) = p(u_A \otimes I_B \psi) \tag{1}$$

where

$$p(\psi) := \mathbb{P}(A_B = a | \text{ state } |\psi\rangle \text{ at } t = t_0 - \epsilon, \text{nonlocal measurement at } t = t_0). \tag{2}$$

This is a restriction because the global unitary transformation applied in the nonlocal measurement need not commute with $u_A \otimes I_B$ or Bob's measurement operators.

4 What Cannot Be Measured: A Paradox

For this Section we will only consider von Neumann measurements. Less restrictive measurements are discussed in Sections 7 and 8.

The question to be answered now is what measurements, if any, satisfy the causality condition of Section 3.3. In this Section we will show that some perfectly reasonable von Neumann measurements do not. In later Sections we will construct procedures that do satisfy the condition.

4.1 Infinite Dimensional System

Causality imposes a severe restriction on what nonlocal measurements are possible. The collapse postulate states there is an instantaneous change in the state, which naïvely can cause violations of causality. We present a paradox here used by Landau and Peierls in 1931 [5], which they claimed proved the impossibility of any instantaneous nonlocal measurement. We only use a single particle, so the condition of Section 3.3 does not apply in this case. However, later the paradox will be modified to a two particle system so the condition does apply.

The paradox can be stated in many forms, but the outcome is always that the possibility of performing the nonlocal measurement on a particular system violates relativistic causality by sending a signal faster than the speed of light, an effect known simply as 'signalling'.

The paradox goes as follows (this form is due to Ref. [2]). Suppose there is a particle localised to some finite region of space A. At a time t_1, an instantaneous measurement of the momentum of the particle is carried out. This, by elementary quantum mechanics, instantaneously changes the state of the particle to an eigenstate of momentum. An eigenstate of momentum has complete uncertainty of position; the particle can be found anywhere in space with equal probability. In particular, if at time $t_1 + \epsilon$ the position of the particle is measured in some region spacelike separated from A, there is a non-zero probability of detecting it there. So by performing this instantaneous momentum measurement we have, with some non-zero probability, succeeded in moving the particle faster than the speed of light. The process is shown in Fig. 3. Clearly this shows information transfer outside the light cone. Landau and Peierls [5] used this to argue that the concept of momentum does not exist at an instant; momentum measurements can only be carried out over times which do not give rise to this paradox. They put forward an uncertainty relation

$$\Delta p \Delta t > \hbar / c$$

which follows from the usual position-momentum relation

$$\Delta x \Delta p > \hbar$$

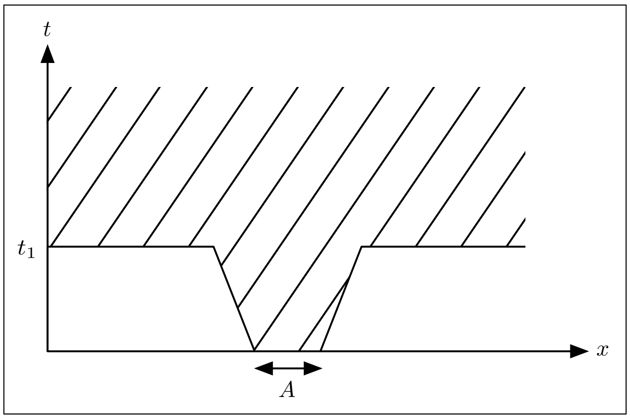


Figure 3: Collapse to a momentum eigenstate. The shaded region indicates where the particle can be found with non-zero probability.

and the condition that the uncertainty in position is limited by the finite speed of propagation,

$$\Delta x < c\Delta t.$$

Landau and Peierls then incorrectly jumped to the conclusion that no nonlocal variables can be measured. The aim of the rest of this essay is to show that the paradox does not apply to all measurements and to show explicitly how some nonlocal measurements can be constructed.

Firstly, following Aharonov and Albert [2], we specialise the above to apply to the case of a spin system of two spin-1/2 particles.

4.2 2×2 Dimensional System

Because of the nonlocality provided by entangled states, the possibility of carrying out some nonlocal measurements can be used to signal between the two spacelike separated particles. In this case, the paradox comes from the fact that for most measurements the collapse to an eigenstate causes a change in the probabilities of outcomes on one particle, depending on the state immediately before the measurement. So an operation carried out on one particle can give different probabilities of outcomes for the other. For example [6, 4], consider a nonlocal measurement with eigenstates

$$\begin{aligned} |\psi_1\rangle &= |\uparrow_z\rangle_A \, |\uparrow_z\rangle_B \,, \\ |\psi_2\rangle &= |\uparrow_z\rangle_A \, |\downarrow_z\rangle_B \,, \\ |\psi_3\rangle &= |\downarrow_z\rangle_A \, |\uparrow_x\rangle_B \,, \\ |\psi_4\rangle &= |\downarrow_z\rangle_A \, |\downarrow_x\rangle_B \end{aligned} \tag{3}$$

where $|\uparrow_z\rangle_A$ means the spin state of Alice's particle with spin aligned along the z axis, $|\downarrow_x\rangle_B$ means the spin state of Bob's particle with spin anti-aligned along the x axis, etc.. Suppose the particles are held by Alice and Bob respectively at spacelike separated points, and suppose the system is initially in the state $|\psi_1\rangle$. If we perform a nonlocal measurement at time t_1, there is no change in state and we verify that the state is $|\psi_1\rangle$. But if, at time $t_1 - \epsilon$, Alice decides to flip her spin then the nonlocal measurement will cause a collapse onto either $|\psi_3\rangle$ or $|\psi_4\rangle$. So if Bob measures his z-axis spin at time $t_1 + \epsilon$, he will obtain spin-down with probability $1/2$. So he can detect if Alice flipped her spin instantaneously, even though they are spacelike separated. This is clearly a violation of causality so this nonlocal measurement does not exist.

5 Necessary Condition for an Instantaneous Nonlocal Measurement

So we have proved by way of a contradiction that not all nonlocal von Neumann measurements are possible. Landau and Peierls thought incorrectly that the paradox of Section 4 applied to *all* nonlocal von Neumann measurements. There, however, exists a class of measurements for which the paradox does not apply. Here we give a condition, due to Popescu and Vaidman [4], for measurements to be in that class. So we now ask what restrictions are placed on the measurement itself so that causality is satisfied. These particular measurements may or may not be realisable; we just show here that they are not immediately unmeasurable by the paradox argument. We will need some other result to show that measurements in this class do in fact exist so some nonlocal variables can be measured. We will show this later by way of an explicit construction of a measurement procedure. The result presented below is independent of the measurement type; it applies to demolition and nondemolition measurements, operator or state specific.

5.1 Condition

We now show that any nonlocal measurement must necessarily erase all local information (apart from degrees of freedom unaffected by the measurement). By this we mean that all possible final states are locally indistinguishable; they have the same reduced density matrices. The measurement we use is a general verification measurement, which means the result can be extended to all measurements (Section 2.2). The only restriction placed on the measurement is that it is reliable; the verification measurement of $|\psi_0\rangle$ will always give "yes" when in state $|\psi_0\rangle$, and always give "no" when in any orthogonal state. This is obviously a necessary requirement for any sensible measurement and indeed we must have reliability to be able to deduce anything about the initial state before the measurement.

So suppose we are verifying the state $|\psi_0\rangle$, which has Schmidt decomposition

$$|\psi_0\rangle = \sum_i \alpha_i \, |i\rangle_A \, |i\rangle_B \, . \tag{4}$$

Let the Hilbert spaces of systems A and B be H_A and H_B respectively. Further let the subspaces spanned by $|i\rangle_A$ and $|i\rangle_B$ be H_{A0} and H_{B0} respectively. Then causality requires that the probability of any outcome for a local measurement performed after this state verification is independent of the initial state. So, using $p(\psi)$ defined in Eqn. 2,

Theorem 1. *If* $|\psi\rangle \in H_A \otimes H_{B0}$ *then* $p(\psi) = p(\psi_0)$

So all outcomes of experiments performed by Bob after the measurement are independent of the initial state i.e. local information has been erased. Equivalently, all the eigenstates that $|\psi\rangle$ collapses onto are locally indistinguishable from $|\psi_0\rangle$.[3]

We must also consider what happens for states outside of $H_A \otimes H_{B0}$. We can think of state verification as simply distinguishing between $|\psi_0\rangle$ and states orthogonal to it. So consider the three disjoint subsets of $H_A \otimes H_B$ shown in Fig. 4. States in any of the three groups can be distinguished from states in any other by local measurements alone, since states in H_{A0} are all orthogonal to states in $H_A - H_{A0}$ and similarly for system B. So a genuine nonlocal measurement is only required to distinguish states in $H_{A0} \otimes H_{B0}$. Theorem 1 says local information must be erased for all initial states in this subspace, so all nonlocal measurements erase local information. Local measurements to distinguish other states do not necessarily erase local information. This is the content of Theorem 2 of Ref. [4]. For rigourous proofs see Ref. [4] also.

[3]Note that the asymmetry between the spaces of particle A and particle B comes from the fact that the local unitary is being performed on particle A.

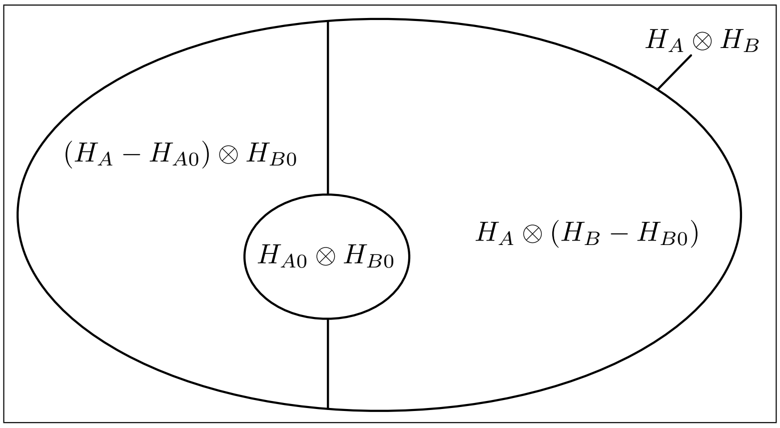


Figure 4: States to be distinguished by a verification measurement

5.2 Bipartite Systems

We now apply the above theorem to the case we are interested in of a bipartite 2×2-dimensional system. We state and prove the following theorem due to Popescu and Vaidman [4].

Theorem 2. *A von Neumann measurement on a 2×2-dimensional bipartite system satisfies the condition of Theorem 1 if and only if it has eigenstates (product states)*

$$\begin{aligned} |\psi_1\rangle &= |\uparrow_z\rangle_A |\uparrow_{z'}\rangle_B \,, \\ |\psi_2\rangle &= |\uparrow_z\rangle_A |\downarrow_{z'}\rangle_B \,, \\ |\psi_3\rangle &= |\downarrow_z\rangle_A |\uparrow_{z'}\rangle_B \,, \\ |\psi_4\rangle &= |\downarrow_z\rangle_A |\downarrow_{z'}\rangle_B \,, \end{aligned} \tag{5a}$$

or (entangled states)

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{2}} \left(|\uparrow_z\rangle_A |\uparrow_{z'}\rangle_B + |\downarrow_z\rangle_A |\downarrow_{z'}\rangle_B \right), \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}} \left(|\uparrow_z\rangle_A |\uparrow_{z'}\rangle_B - |\downarrow_z\rangle_A |\downarrow_{z'}\rangle_B \right), \\ |\psi_3\rangle &= \frac{1}{\sqrt{2}} \left(|\uparrow_z\rangle_A |\downarrow_{z'}\rangle_B + |\downarrow_z\rangle_A |\uparrow_{z'}\rangle_B \right), \\ |\psi_4\rangle &= \frac{1}{\sqrt{2}} \left(|\uparrow_z\rangle_A |\downarrow_{z'}\rangle_B - |\downarrow_z\rangle_A |\uparrow_{z'}\rangle_B \right). \end{aligned} \tag{5b}$$

The z and z' axes are labelled this way because they do not have to be the same for both particles. It is important however, as we shall see, that they are the same direction amongst all the states.

Before giving the proof, we note that the above says essentially only one nonlocal operator is measurable in the von Neumann sense. The fact that virtually all operators are unmeasurable can be seen from the two restrictions imposed on the measurement. Firstly, we require the final state to be an eigenstate of the operator with systems already in an eigenstate being undisturbed. Yet, causality requires the final state to be locally independent of the initial state. Simultaneously satisfying these conditions greatly restricts the possible measurements.

Proof of Theorem 2. Firstly show the states of Theorem 2 satisfy the condition of Theorem 1.

Product states. It is clear in this case that any local unitary transform Alice applies cannot cause Bob's spin to flip. Indeed, Alice can only change the final state to any $\alpha \left|\psi_1\right\rangle + \beta \left|\psi_3\right\rangle$ or $\alpha \left|\psi_2\right\rangle + \beta \left|\psi_4\right\rangle$. These are locally indistinguishable for Bob.

Entangled states. Since any entangled state of this bipartite system has Schmidt decomposition with exactly two terms, the states in the decomposition span the local spaces, so we have $H_{A0} = H_A$ and $H_{B0} = H_B$. So the condition we must satisfy is that for *any* initial state, the final states (i.e. eigenstates) are locally indistinguishable. Clearly the entangled states above have the same reduced density matrices so this is satisfied.

Now for the reverse part. We must show that these are the only states allowed.

Product states. The most general form of four orthogonal product states is

$$\begin{aligned} \left|\psi_1\right\rangle &= \left|\uparrow_z\right\rangle_A \left|\uparrow_{z'}\right\rangle_B , \\ \left|\psi_2\right\rangle &= \left|\uparrow_{z''}\right\rangle_A \left|\downarrow_{z'}\right\rangle_B , \\ \left|\psi_3\right\rangle &= \left|\downarrow_z\right\rangle_A \left|\uparrow_{z'}\right\rangle_B , \\ \left|\psi_4\right\rangle &= \left|\downarrow_{z''}\right\rangle_A \left|\downarrow_{z'}\right\rangle_B . \end{aligned} \tag{6}$$

Alice can perform bit flips, so we require $\left|\uparrow_z\right\rangle_A$ to be locally indistinguishable from $\left|\uparrow_{z''}\right\rangle_A$, which can only be true if $z = z''$. So we have reduced the general form to the states given.

Entangled states. As before, all eigenstates must be locally indistinguishable. Let $\left|\psi_1\right\rangle = \alpha \left|\uparrow_z\right\rangle_A \left|\uparrow_{z'}\right\rangle_B + \beta \left|\downarrow_z\right\rangle_A \left|\downarrow_{z'}\right\rangle_B$, which follows from Schmidt decomposition. We can easily see that there are no states orthogonal to $\left|\psi_1\right\rangle$ and locally indistinguishable from it unless $|\alpha| = |\beta|$. So for $\left|\psi_1\right\rangle$ to be an eigenstate it must be a maximally entangled state. So all eigenstates are maximally entangled states. It can be shown that (e.g. [4]), by making a choice of local bases, we can write the maximally entangled states in the above form. $\square$

6 Is Anything Measurable? An Explicit Construction

Now we know that there is only essentially one nonlocal von Neumann measurement, we just need to show that it is indeed realisable. So we now give an explicit method of how to measure one nonlocal variable, due to Aharonov et al [3]. We will then generalise this construction slightly and show explicitly that it satisfies the necessary condition above. We will then have that, in fact, all von Neumann measurements that satisfy this condition can be composed of measurements of this type. So a variable is operator specific measurable if and only if it is of this generalised type.

We construct a method to measure the sum of two local variables with operators A_A and A_B acting on the two respective subspaces alone. So we aim to verify that $A_A \otimes I_B + I_A \otimes A_B = a$ for some constant a. Without loss of generality, take a to be 0. The method below is a nonlocal measurement because we never know the values of A_A and A_B individually. Doing so would necessarily destroy any entanglement between the particles.

The whole method relies upon entangling the measuring devices beforehand. This can be done as far in the past as we like, and the devices can be together at this time to interact locally. We then separate them to be at the points of interest. So let the measuring devices have canonical coordinates q_A and q_B respectively, with canonical conjugate momenta π_A and π_B respectively. We can think of the coordinate as a position of a needle on a dial, and the conjugate momentum the momentum of the needle. We set the initial (entangled) state to[4]

$$q_A + q_B = 0, \pi_A - \pi_B = 0 \tag{7}$$

[4]Note that, following Aharonov and Rohrlich [7], we have swapped the roles of q and π as it is more natural to consider measurements of the canonical coordinate rather than measurements of the conjugate momentum.

Now the measuring devices interact locally with the separate systems. The interacting Hamiltonian is

$$\begin{aligned} H_{int} &= g(t)(\pi_A A_A \otimes I_B + \pi_B I_A \otimes A_B) \\ &\equiv H_{A,int} + H_{B,int} \end{aligned} \tag{8}$$

where $g(t)$ is nonzero for a time ϵ and $H_{A,int}$ and $H_{B,int}$ have the obvious definitions.

So the Heisenberg operators satisfy

$$\begin{aligned} \dot{q}_A &= i[H_{int}, q_A] \\ &= ig(t) A_A \otimes I_B [\pi_A, q_A] \\ &= g(t) A_A \otimes I_B \end{aligned} \tag{9}$$

Similarly $\dot{q}_B = g(t) I_A \otimes A_B$.

We can now solve these, taking ϵ small enough so A_A and A_B do not change by their own dynamics. (We do not assume they commute with H_{int}; there may be extra terms in the Hamiltonian neglected above.) We get

$$(q_A + q_B)|_{t > t_0 + \epsilon} = (A_A \otimes I_B + I_A \otimes A_B)|_{t = t_0} \tag{10}$$

Thus a local measurement of q_A and q_B completes the measurement. Then we use classical communication between Alice and Bob to communicate the results. Since all local operations take an arbitrarily small amount of time the measurement is instantaneous in the sense of Sec. 3.1.

6.1 Final State

We must now work out the final state after the measurement. For the state we are measuring, with $A_A \otimes I_B + I_A \otimes A_B = 0$,

$$\begin{aligned} H_{int} \left| A_A \otimes I_B + I_A \otimes A_B = 0 \right\rangle &= g(t) \pi_A (A_A \otimes I_B + I_A \otimes A_B) \left| A_A \otimes I_B + I_A \otimes A_B = 0 \right\rangle \\ &= 0 \end{aligned} \tag{11}$$

since $\pi_A = \pi_B$ initially and π_A and π_B are unchanged (they commute with H_{int}). So this is a non-demolition experiment and, in fact, for any eigenstate the system is undisturbed since the Hamiltonian acts only on the measuring device. So we have a standard von Neumann measurement.

6.2 Modular Sum

It turns out that the above measurement is not quite general enough to carry out every measurement. We now extend it to measure variables modulo some constant x. This means we only know the eigenvalues modulo x; this lack of information gives less restrictive eigenstates.

If instead of setting $q_A + q_B = 0$ initially, we set $q_A + q_B \mod x = 0$. With the same Hamiltonian, the evolution gives us the value of $A_A \otimes I_B + I_A \otimes A_B \mod x$, which is what we seek.

6.3 Erasure of Local Information

We now easily show that the necessary condition of Theorem 2 is satisfied by this measurement. Indeed it must if quantum mechanics is to not violate special relativity. Since we have already shown in Sec. 3.2 that no local operations in quantum mechanics can send superluminal signals, we already know that this measurement, consisting only of local operations, must satisfy the condition but it is reassuring to check.

It is clear that we can diagonalise $A_A \otimes I_B + I_A \otimes A_B$ using local unitary transforms. Let u_A be the unitary operator that diagonalises A_A such that $u_A^\dagger A_A u_A = D_A$ where D_A is diagonal. Define u_B and D_B similarly. So we have

$$\begin{aligned} u_A^\dagger \otimes u_B^\dagger (A_A \otimes I_B + I_A \otimes A_B) u_A \otimes u_B &= u_A^\dagger A_A u_A \otimes I_B + I_A \otimes u_B^\dagger A_B u_B \\ &= D_A \otimes I_B + I_A \otimes D_B. \end{aligned} \tag{12}$$

So we can always write $A_A \otimes I_B + I_A \otimes A_B$ in a diagonal form, using *local* unitary transforms. So the eigenvectors are the product states or linear combinations if there are degenerate eigenvalues. Either way, the collapse does not violate causality since we do not cause a collapse onto any state disallowed by Theorem 2.

On the other hand, if any operator B is locally diagonisable then an operator A of the sum form above can be constructed to have the same eigenvectors and same distinguishability of eigenvalues.[5] So this method measures precisely those operators that are locally diagonisable. It is very important to note that simple projective measurements carried out on both particles can also measure any locally diagonisable operator, but should there be any degeneracy, so an entangled state is an eigenstate, the projective measurements will destroy this. This is because it is always possible to distinguish the four product states using a projective measurement, but using this nonlocal method degeneracy will allow us to remain ignorant of this distinction so entanglement is preserved. Note this is equivalent to the fact that the nonlocal measurement above does *not* determine A_A and A_B separately, whereas the local projective measurement does. This is crucial not only to give the correct final state, but the entanglement must be preserved so a further measurement can be carried out to actually distinguish the entangled states (see Section 6.4).

6.4 Full measurement procedure

A single measurement of the above type is not enough for most measurements. We in fact make two nonlocal measurements, performing local unitary operations in between. So the procedure is

1. Measure $A = A_A \otimes I_B + I_A \otimes A_B \mod x$;
2. Perform local unitary operations u_A and u_B;
3. Measure $B = B_A \otimes I_B + I_A \otimes B_B \mod x$.

So we require two entangled measuring devices. Note all procedures above can be completed in an arbitrarily small amount of time so the complete measurement procedure is instantaneous. Further, to return the system to the initial state we can apply $u_A^\dagger$ and $u_B^\dagger$ after step 3.

Finally we are in a position to show explicitly how the measurement can be performed. We firstly distinguish between the two groups of eigenvectors $|\psi_1\rangle$, $|\psi_2\rangle$ and $|\psi_3\rangle$, $|\psi_4\rangle$. Then we apply a local unitary operator, and then distinguish within the groups. The operator A we measure is, in the $H_A \otimes H_B$ basis,

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mod 2 \tag{13}$$

[5] We are not free to choose any eigenvalues for operators of the form $A_A \otimes I_B + I_A \otimes A_B$, but we can choose different values with the same degeneracy of any other operator to make an effectively equivalent operator.

which is of the above form since we are measuring $\mod 2$. In terms of the local measurement operators, we have

$$A_A = \frac{1}{2}\sigma_{z_A} \tag{14}$$

$$A_B = \frac{1}{2}\sigma_{z_B}. \tag{15}$$

The first measurement distinguishes $|\psi_1\rangle$, $|\psi_2\rangle$, with eigenvalues 1, from $|\psi_3\rangle$, $|\psi_4\rangle$, with eigenvalues 0. Now we apply the local unitary operator (the Hadamard transform)

$$T = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{16}$$

to both particles giving the transformation

$$|\psi_1\rangle \to |\psi_1\rangle\,, |\psi_2\rangle \to |\psi_3\rangle\,, |\psi_3\rangle \to |\psi_2\rangle\,, |\psi_4\rangle \to |\psi_4\rangle\,. \tag{17}$$

So now measure the above operator again and we have succeeded in distinguishing the states. This procedure is shown graphically in Fig. 5.

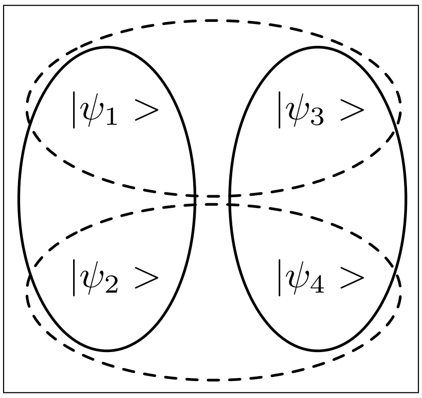


Figure 5: Groups of states distinguished in two measurements. States in different solid line groups are distinguished by the first measurement; states in different dotted line groups by the second.

So we have finally shown that the modular sum measurement procedure is the only method we need. So we have

Theorem 3. *A nonlocal operator on a* 2×2 *dimensional system is von Neumann measurable instantaneously if and only if it can be measured in the form above.*

Proof of Theorem 3. By construction, we can, in principle, perform the above measurement method. By Theorem 2, any measurement has one of two forms of eigenstates, both of which this procedure can measure. $\square$

Lastly, we note that there may exist alternative measurement methods, but they must have the same eigenstates. It is thought [3], however, that no such different method exists.

7 State Specific Measurements

We now turn to a less restrictive measurement and consequently find more can be measured. Here we are not interested in determining the eigenvalue of some operator (operator specific measurement), but are only asking the question whether we have one particular state or something orthogonal to it. We further add the restriction that this is a nondemolition measurement: if we are verifying the state $|\psi_0\rangle$, and it is in this state initially, the system is undisturbed. On the other hand, if the initial state is orthogonal to $|\psi_0\rangle$, the system remains in some orthogonal state, although not necessarily the original state. For linear combinations of these cases, the projection onto $|\psi_0\rangle$ or the orthogonal space is taken, depending on the result. This is very different to the von Neumann measurement above, because we do not require that eigenstates other than $|\psi_0\rangle$ remain unaltered during the procedure. This is why more operators become measurable in this sense; the procedures that verify all the eigenstates may exist, but they are incompatible with each other so some eigenstates are disturbed so the operator cannot be measured in an operator specific sense.

We can use the constructions above to perform the verification. The final state restrictions of Theorem 2 mean that only maximally entangled states or product states can be verified in this way, as for operator specific measurements. However, the reason why more operators are measurable here is that the operator can have any mixture of these states, rather than all entangled or all unentangled. We can use the measurement procedure of Section 6.4 to verify the maximally entangled eigenstates, and local projections to verify the unentangled states. For the former, the final states will all be maximally entangled states; for the latter, the final states will be the product states. As an example, we show how the total spin eigenstates

$$\begin{aligned}
|J=1, m_J=1\rangle &= \left(|{\uparrow_z}\rangle_A |{\uparrow_z}\rangle_B\right), \\
|J=1, m_J=0\rangle &= \frac{1}{\sqrt{2}}\left(|{\uparrow_z}\rangle_A |{\downarrow_z}\rangle_B + |{\downarrow_z}\rangle_A |{\uparrow_z}\rangle_B\right), \\
|J=1, m_J=-1\rangle &= \left(|{\downarrow_z}\rangle_A |{\downarrow_z}\rangle_B\right), \\
|J=0, m_J=0\rangle &= \frac{1}{\sqrt{2}}\left(|{\uparrow_z}\rangle_A |{\downarrow_z}\rangle_B - |{\downarrow_z}\rangle_A |{\uparrow_z}\rangle_B\right)
\end{aligned} \tag{18}$$

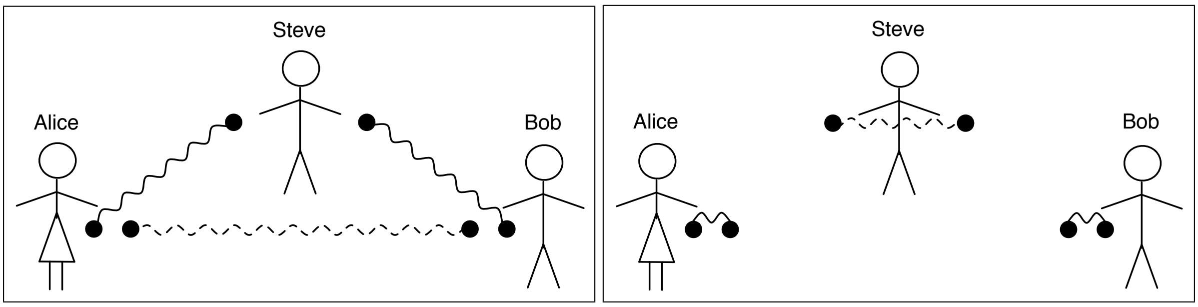
can be verified. We can determine if we are in state $|J=1, m_J=0\rangle$ or $|J=0, m_J=0\rangle$ by performing the nonlocal Bell measurement of Section 6.4. This will leave both of these states undisturbed but $|J=1, m_J=1\rangle$ and $|J=1, m_J=-1\rangle$ will end up in one of the other two Bell states. Further, we cannot tell which of the $m_J \neq 0$ states the system was in initially. Similarly, to verify $|J=1, m_J=1\rangle$ or $|J=1, m_J=-1\rangle$ we can just perform a local projective measurement. Clearly these states remain unchanged and the two other states end up in one of the product states and cannot then be distinguished.

8 Demolition Operator Specific Measurements

We require, in the case of operator specific measurements, the final state to be the eigenstate of the operator corresponding to the resultant eigenvalue. This gave the restriction on what could be measured when coupled with relativistic causality. Now we relax this requirement; a measurement is still meaningful and useful if it does not prepare the eigenstate. We can still find the eigenvalue, which tells us about the state beforehand. We can think of the final state preparation as a process separate to measurement. With this definition of measurement, which we call here a demolition operator specific measurement, all 2×2-dimensional bipartite operators become measurable.

8.1 Exchange Measurement

Any nonlocal operator can be measured using an exchange measurement [3]. Here we use teleportation [8], as suggested by Popescu and Vaidman [4], to bring the quantum states together so a nonlocal collective measurement can be made. As above, we assume any collective measurement can be made when the particles are in the same small region.



(a) Initial State

(b) Final State

Figure 6: (a) The initial configuration, before the exchange measurement. (b) The final configuration, after the measurement. Black circles represent particles; solid wavy lines indicate particles are maximally entangled; dotted wavy lines indicate particles may be entangled by any amount.

So the procedure is as follows

- Rather than giving Alice and Bob an entangled measuring device by *Steve*, they are each given one particle of an EPR pair. The other particle of each pair is kept by Steve, so both Alice and Bob share an entangled pair with Steve. See Fig. 6(a).
- Using quantum teleportation, both Alice and Bob send their state to Steve. Teleportation consists of local interactions and measurements, together with classical communication of two bits each.
- Now Steve's two particles are in the states originally held by Alice and Bob[6]. Since they are in the same location, Steve can perform a collective measurement of any operator to determine the state.
- Steve then communicates the measurement result to Alice and Bob.

There are a number of important remarks. Firstly, the final state of Alice and Bob's particles is one of the Bell states. The other member of the pair is the respective particle they were given by Steve (See Fig. 6(b)). The Bell state is randomly chosen with equal probabilities and Alice and Bob's final states are independent. So the final state is completely independent of the initial state; local information is erased.

Secondly, the procedure is not instantaneous in the sense of Sec. 3.1, even though the interaction with the system (Alice and Bob's original particles) is instantaneous. The teleportation 'freezes' a copy of this state, encoded by Steve's particles and the classical bits being sent to Steve. So the collapse of Alice and Bob's particles is complete and no interaction with them can change the result. However, the result is not classically recorded until Steve makes his measurement. He cannot do this until he receives the results of the local measurements from Alice and Bob. In fact, the outcome does not exist until this happens [6]. So we cannot say that this gives the variable the status of an observable.

[6]Note the entanglement is preserved since teleportation is linear

8.2 Another Method

There, however, exist measurement procedures that can measure any variable instantaneously using our definition. We will discuss a method due to Groisman and Reznik [9]. Vaidman has also presented an alternative [6].

Like the exchange method, we require entanglement resources, although in this case the pairs are shared between Alice and Bob, rather than some other figure. Again, Alice and Bob each perform local interactions and local measurements, taking an arbitrarily small amount of time. Then the results of the local measurements are all that are needed to determine the overall result, so this procedure does measure the variable instantaneously.

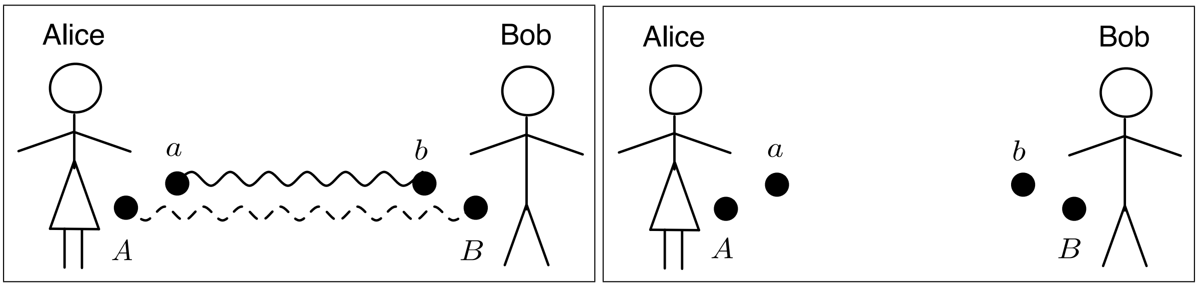
We will illustrate the method for the twisted product basis only. This demonstrates all the necessary ingredients for the general case of any basis.

So we are measuring an operator with eigenvectors (note this is the same as Eqn. 3)

$$\begin{aligned} |\psi_1\rangle_{AB} &= |\uparrow_z\rangle_A |\uparrow_{z'}\rangle_B , \\ |\psi_2\rangle_{AB} &= |\uparrow_z\rangle_A |\downarrow_{z'}\rangle_B , \\ |\psi_3\rangle_{AB} &= |\downarrow_z\rangle_A \frac{1}{\sqrt{2}} \left(|\uparrow_{z'}\rangle_B + |\downarrow_{z'}\rangle_B\right) , \\ |\psi_4\rangle_{AB} &= |\downarrow_z\rangle_A \frac{1}{\sqrt{2}} \left(|\uparrow_{z'}\rangle_B - |\downarrow_{z'}\rangle_B\right) . \end{aligned} \tag{19}$$

Alice and Bob use one shared EPR pair to perform a 'controlled rotation' on Bob's qubit. If Alice's qubit has spin down, we want to rotate Bob's by $\pi/2$ so we get the computational, i.e. product state, basis. Below this transformation to the computational basis is described.

Let the joint state of Alice and Bob's particles be $|\psi\rangle_{AB}$, and choose the shared EPR pair to have state $\frac{1}{\sqrt{2}} \left(|\uparrow_z\rangle_a \otimes |\uparrow_{z'}\rangle_b + |\downarrow_z\rangle_a \otimes |\downarrow_{z'}\rangle_b\right)$, where lower case a refers to Alice's particle and b for Bob's.



(a) Initial State

(b) Final State

Figure 7: (a) The initial configuration, before the measurement. (b) The final configuration, after the measurement. Black circles represent particles; solid wavy lines indicate particles are maximally entangled; dotted wavy lines indicate particles may be entangled by any amount. Note in the example used here there is no entanglement between the particles being measured, but in the general case there can be any amount.

The algorithm is as follows:

1. Initial state is (Fig. 7(a))

$$\frac{1}{\sqrt{2}} \left(|\uparrow_z\rangle_a \otimes |\uparrow_{z'}\rangle_b + |\downarrow_z\rangle_a \otimes |\downarrow_{z'}\rangle_b\right) \otimes |\psi\rangle_{AB} . \tag{20}$$

2. Bob performs a controlled-NOT operation with particle b as the control and $\sigma_{y'_b}$ as the NOT operation giving state

$$\frac{1}{\sqrt{2}}\left(|{\uparrow_z}\rangle_a \otimes |{\uparrow_{z'}}\rangle_b \otimes I_B + |{\downarrow_z}\rangle_a \otimes |{\downarrow_{z'}}\rangle_b \otimes \sigma_{y'_b}\right)|\psi\rangle_{AB}\,. \tag{21}$$

3. Bob measures $\sigma_{x'_b}$, with result $v(\sigma_{x'_b})$ giving the state (leaving out particle b)

$$\frac{1}{\sqrt{2}}\left(|{\uparrow_z}\rangle_a \otimes I_B + v(\sigma_{x'_b})\,|{\downarrow_z}\rangle_a \otimes \sigma_{y'_b}\right)|\psi\rangle_{AB}\,. \tag{22}$$

4. Alice measures σ_{z_A}. If $v(\sigma_{z_A}) = 1$

4.1. Alice measures σ_{z_a} so Bob's particle is in state

$$\left(\frac{1+v(\sigma_{z_a})}{2} I_B + v(\sigma_{x'_b})\frac{1-v(\sigma_{z_a})}{2}\sigma_{y'_b}\right)|\psi\rangle_B \tag{23}$$

Clearly then for $v(\sigma_{z_a}) = 1$, $|\psi\rangle_B$ is unchanged; for $v(\sigma_{z_a}) = -1$, $|\psi\rangle_B$ is flipped. So the transformation is

$$|\psi_1\rangle_{AB}\,, |\psi_2\rangle_{AB} \to \begin{cases} |{\uparrow_z}\rangle_A\,|{\uparrow_{z'}}\rangle_B\,, |{\uparrow_z}\rangle_A\,|{\downarrow_{z'}}\rangle_B & v(\sigma_{z_a}) = +1, \\ |{\uparrow_z}\rangle_A\,|{\downarrow_{z'}}\rangle_B\,, |{\uparrow_z}\rangle_A\,|{\uparrow_{z'}}\rangle_B & v(\sigma_{z_a}) = -1. \end{cases} \tag{24}$$

else (so we have $v(\sigma_{z_A}) = -1$)

4.2. Alice acts with $e^{i\pi\sigma_{x_a}/4}$ on her qubit a. Since $v(\sigma_{z_A}) = -1$, we have $|\psi_B\rangle = \frac{1}{\sqrt{2}}\left(|{\uparrow_{z'}}\rangle_B \pm |{\downarrow_{z'}}\rangle_B\right)$. We can now see that this operation performed by Alice is equivalent to a rotation of Bob's qubit. The resultant state is

$$\begin{aligned}\frac{1}{2\sqrt{2}}(|{\uparrow_z}\rangle_a \otimes [(1 \pm v(\sigma_{x'_b}))\,|{\uparrow_{z'}}\rangle_B + (\pm 1 - v(\sigma_{x'_b}))\,|{\downarrow_{z'}}\rangle_B] + \\ |{\downarrow_z}\rangle_a \otimes i[(1 \mp v(\sigma_{x'_b}))\,|{\uparrow_{z'}}\rangle_B + (\pm 1 + v(\sigma_{x'_b}))\,|{\downarrow_{z'}}\rangle_B])\end{aligned} \tag{25}$$

4.3. Now Alice measures σ_{z_a}. So the transformation here is

$$|\psi_3\rangle_{AB}\,, |\psi_4\rangle_{AB} \to \begin{cases} |{\downarrow_z}\rangle_A\,|{\uparrow_{z'}}\rangle_B\,, |{\downarrow_z}\rangle_A\,|{\downarrow_{z'}}\rangle_B & v(\sigma_{z_a})v(\sigma_{x'_b}) = +1, \\ |{\downarrow_z}\rangle_A\,|{\downarrow_{z'}}\rangle_B\,, |{\downarrow_z}\rangle_A\,|{\uparrow_{z'}}\rangle_B & v(\sigma_{z_a})v(\sigma_{x'_b}) = -1. \end{cases} \tag{26}$$

So now all that we have remaining to do is for Bob to measure $\sigma_{z'_B}$. Now Alice sends Bob $v(\sigma_{z_a})$ and $v(\sigma_{z_A})$; Bob sends Alice $v(\sigma_{x'_b})$ and $v(\sigma_{z'_B})$. From the two classical bits received, both Alice and Bob can work out which transformation was realised and which product state they ended up in so can infer the original state. So the classical bits resulting from the four measurements specify the result so the measurement procedure is instantaneous, as claimed above. Note also that all entanglement is destroyed in the process (see Fig. 7(b)).

We can easily see that this procedure satisfies the necessary condition imposed by causality. Because each measurement outcome is equiprobable, the map is non-deterministic. From Bob's point of view, $v(\sigma_{z_a})$ is $+1$ or -1 with equal probability, so his two measurement results give no information at all about the initial state of Alice's particle. Similarly Alice can obtain no information about the initial state of Bob's particle, because her state remains unchanged throughout the whole process.

9 Conclusions and Summary

We have seen how measurement procedures can be constructed to measure any nonlocal variable of a 2×2-dimensional bipartite system instantaneously. So all nonlocal variables of this system can be regarded as observables. However, only a small number of nonlocal variables can be measured instantaneously if we constrain the final state to be an eigenstate.

We have only considered bipartite systems of two dimensions each. For more particles and more dimensions the situation is more complicated and we have not attempted to generalise our results as has been done by Vaidman [6]. Vaidman has shown that within this general framework *all* nonlocal variables are observable.

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