Using the Runge-Lenz vector to derive the orbit in a $1/r$ potential

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Here we find a conserved quantity, the Runge-Lenz Vector, in any orbit in a $1/r$ potential and use this to show the orbit is elliptical or, in general, any conic section.

First define the specific angular momentum $\boldsymbol{h}$

$$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}}$$

and the force law for an inverse square central force

$$\ddot{\boldsymbol{r}} = -\frac{k\boldsymbol{r}}{r^3}$$

where k is a constant e.g. for gravity, $k = GM$.

Now prove $\boldsymbol{h}$ is conserved

$$\begin{aligned} \dot{\boldsymbol{h}} &= \dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}} \\ &= \boldsymbol{0} - \frac{k}{r^3}\boldsymbol{r} \times \boldsymbol{r} \\ &= \boldsymbol{0} \end{aligned}$$

Secondly, find the time derivative of $\boldsymbol{r}/r$:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\boldsymbol{r}}{r}\right) = \frac{1}{r^2}\left(r\,\dot{\boldsymbol{r}} - \dot{r}\,\boldsymbol{r}\right)$$

but

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\boldsymbol{r}.\boldsymbol{r}\right) = \frac{\mathrm{d}}{\mathrm{dt}}\left(r^2\right)$$

so

$$2\dot{\boldsymbol{r}}.\boldsymbol{r} = 2r\dot{r}$$

giving

$$\dot{r} = \frac{\dot{\boldsymbol{r}}.\boldsymbol{r}}{r}$$

so

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\boldsymbol{r}}{r}\right) = \frac{1}{r^3}\left(r^2\dot{\boldsymbol{r}} - \dot{\boldsymbol{r}}.\boldsymbol{r}\,\boldsymbol{r}\right)$$

Now consider $\frac{\mathrm{d}}{\mathrm{dt}}\left(\dot{\boldsymbol{r}} \times \boldsymbol{h}\right)$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}}\left(\dot{\boldsymbol{r}} \times \boldsymbol{h}\right) &= \ddot{\boldsymbol{r}} \times \boldsymbol{h} + \dot{\boldsymbol{r}} \times \dot{\boldsymbol{h}} \\ &= \ddot{\boldsymbol{r}} \times \left(\boldsymbol{r} \times \dot{\boldsymbol{r}}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{k}{r^3}\boldsymbol{r} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}}) \\
&= -\frac{k}{r^3}\left(\boldsymbol{r}.\dot{\boldsymbol{r}}\,\boldsymbol{r} - r^2\dot{\boldsymbol{r}}\right) \\
&= \frac{k}{r^3}\left(r^2\dot{\boldsymbol{r}} - \boldsymbol{r}.\dot{\boldsymbol{r}}\,\boldsymbol{r}\right) \\
&= k\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{\boldsymbol{r}}{r}\right)
\end{aligned}
$$

Therefore

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\dot{\boldsymbol{r}} \times \boldsymbol{h} - k\,\boldsymbol{r}/r\right) = \boldsymbol{0}$$

This gives the Runge-Lentz vector $\boldsymbol{A}$

$$\dot{\boldsymbol{r}} \times \boldsymbol{h} - k\,\boldsymbol{r}/r \equiv \boldsymbol{A} \text{ is conserved.}$$

Now consider $\boldsymbol{A}.\boldsymbol{r}$:

$$
\begin{aligned}
\boldsymbol{A}.\boldsymbol{r} &= Ar\cos\theta \\
&= \boldsymbol{r}.\left(\dot{\boldsymbol{r}} \times (\boldsymbol{r} \times \dot{\boldsymbol{r}})\right) - kr \\
&= (\boldsymbol{r} \times \dot{\boldsymbol{r}})\,.\,(\boldsymbol{r} \times \dot{\boldsymbol{r}}) - kr \\
&= h^2 - kr
\end{aligned}
$$

So

$$
\begin{aligned}
r &= \frac{h^2}{k + A\cos\theta} \\
&= \frac{\frac{h^2}{k}}{1 + \frac{A}{k}\cos\theta}
\end{aligned}
$$

This is a conic section because h and A are constant. The semi-latus rectum is h^2/k and eccentricity A/k with the 'sun' at one focus. A special case is when $A/k = 0$, a circular orbit. For bound orbits, $0 \leq A/k < 1$, which gives an ellipse.